Problem 1

If $a_n = \dfrac{(-3)^n x^n}{n^{3/2}}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{(n+1)^{3/2}}\cdot\frac{n^{3/2}}{(-3)^n x^n}\right| = \lim_{n\to\infty}\left|-3x\left(\frac{n}{n+1}\right)^{3/2}\right| = 3\,|x|\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)^{3/2}$$
$$= 3\,|x|\,(1) = 3\,|x|$$

By the Ratio Test, the series $\sum_{n=1}^{\infty}\frac{(-3)^n}{n\sqrt{n}}\,x^n$ converges when $3\,|x| < 1 \quad\Leftrightarrow\quad |x| < \frac{1}{3}$, so $R = \frac{1}{3}$. When $x = \frac{1}{3}$, the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{3/2}}$ converges by the Alternating Series Test. When $x = -\frac{1}{3}$, the series $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ is a convergent p-series $\left(p = \frac{3}{2} > 1\right)$. Thus, the interval of convergence is $\left[-\frac{1}{3}, \frac{1}{3}\right]$.

Problem 2

$\sum_{n=2}^{\infty}(1+c)^{-n}$ is a geometric series with $a=(1+c)^{-2}$ and $r=(1+c)^{-1}$, so the series converges when

$\left|(1+c)^{-1}\right|<1 \quad \Leftrightarrow \quad |1+c|>1 \quad \Leftrightarrow \quad 1+c>1$ or $1+c<-1 \quad \Leftrightarrow \quad c>0$ or $c<-2$. We calculate the sum of the

series and set it equal to 2: $\dfrac{(1+c)^{-2}}{1-(1+c)^{-1}}=2 \quad \Leftrightarrow \quad \left(\dfrac{1}{1+c}\right)^2=2-2\left(\dfrac{1}{1+c}\right) \quad \Leftrightarrow \quad 1=2(1+c)^2-2(1+c) \quad \Leftrightarrow$

$2c^2+2c-1=0 \quad \Leftrightarrow \quad c=\frac{-2\pm\sqrt{12}}{4}=\frac{\pm\sqrt{3}-1}{2}$. However, the negative root is inadmissible because $-2<\frac{-\sqrt{3}-1}{2}<0$.

So $c=\frac{\sqrt{3}-1}{2}$.

Problem 3

The auxiliary equation is $ar^2+br+c=0$.

If $b^2-4ac>0$, then any solution is of the form

$$y(x)=c_1e^{r_1x}+c_2e^{r_2x} \text{ where } r_1=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } r_2=\frac{-b-\sqrt{b^2-4ac}}{2a} .$$

But a , b , and c are all positive so both r_1 and r_2 are negative and $y(x)=0$

If $b^2-4ac=0$, then any solution is of the form

$y(x)=c_1e^{rx}+c_2xe^{rx}$ where $r=-b/(2a)<0$ since a , b are positive. Hence $y(x)=0$.

if $b^2-4ac<0$

then any solution is of the form $y(x)=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$ where $\alpha=-b/(2a)<0$ since a and b are positive. Thus $y(x)=0$.

Problem 4

(a) *Case 1* ($\lambda = 0$): $y'' + \lambda y = 0 \quad \Rightarrow \quad y'' = 0$ which has an auxiliary equation $r^2 = 0 \quad \Rightarrow \quad r = 0 \quad \Rightarrow \quad y = c_1 + c_2 x$

where $y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1$ and $0 = y(L) = c_2 L \quad \Rightarrow \quad c_1 = c_2 = 0$. Thus $y = 0$.

Case 2 ($\lambda < 0$): $y'' + \lambda y = 0$ has auxiliary equation $r^2 = -\lambda \quad \Rightarrow \quad r = \pm\sqrt{-\lambda}$ [distinct and real since $\lambda < 0$] $\Rightarrow$

$y = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$ where $y(0) = 0$ and $y(L) = 0$. Thus $0 = y(0) = c_1 + c_2$ $(*)$ and

$0 = y(L) = c_1 e^{\sqrt{-\lambda}L} + c_2 e^{-\sqrt{-\lambda}L}$ $(\dagger)$.

Multiplying $(*)$ by $e^{\sqrt{-\lambda}L}$ and subtracting $(\dagger)$ gives $c_2\left(e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L}\right) = 0 \quad \Rightarrow \quad c_2 = 0$ and thus $c_1 = 0$ from $(*)$.

Thus $y = 0$ for the cases $\lambda = 0$ and $\lambda < 0$.

(b) $y'' + \lambda y = 0$ has an auxiliary equation $r^2 + \lambda = 0 \quad \Rightarrow \quad r = \pm i\sqrt{\lambda} \quad \Rightarrow \quad y = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$ where

$y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1$ and $0 = y(L) = c_2 \sin\sqrt{\lambda}L$ since $c_1 = 0$. Since we cannot have a trivial

solution, $c_2 \neq 0$ and thus $\sin\sqrt{\lambda}L = 0 \quad \Rightarrow \quad \sqrt{\lambda}L = n\pi$ where n is an integer $\quad \Rightarrow \quad \lambda = n^2\pi^2/L^2$ and

$y = c_2 \sin(n\pi x/L)$ where n is an integer.

Problem 5

$$\mathcal{F}_x\,[\cos(2\,\pi\,k_0\,x)]\,(k) = \int_{-\infty}^{\infty} e^{-2\,\pi\,i\,k\,x}\left(\frac{e^{2\,\pi\,i\,k_0\,x} + e^{-2\,\pi\,i\,k_0\,x}}{2}\right) d\,x$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\left[e^{-2\,\pi\,i\,(k-k_0)\,x} + e^{-2\,\pi\,i\,(k+k_0)\,x}\right] d\,x$$

$$= \frac{1}{2}\,[\delta\,(k-k_0) + \delta\,(k+k_0)],$$

Problem 6

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}. \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx.$$

Assuming that the solution can be represented as a Fourier series expansion

$$y = \sum_{n=-\infty}^{\infty} y_n e^{inx}, \longrightarrow y' = \sum_{n=-\infty}^{\infty} i n y_n e^{inx}.$$

Substituting this into the differential equation, we get

$$\sum_{n=-\infty}^{\infty} i n y_n e^{inx} + k\sum_{n=-\infty}^{\infty} y_n e^{inx} = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

Since this equation is valid for all *n*, we obtain

$$i n y_n + k y_n = c_n \quad \text{or} \quad y_n = \frac{c_n}{in + k}.$$

$$\boxed{y(x) = \sum_{n=-\infty}^{\infty} \frac{c_n}{in + k} e^{inx}.}$$